- Fundamental question: How do we represent the partial derivatives of a vector-valued function, especially one that has an input of a vector of *n* variables but an output of a vector of *m* variables $(x_1, x_2, ..., x_m) = (f_1(u_1, u_2, ..., u_n), f_2(u_1, u_2, ..., u_n), ..., f_m(u_1, u_2, ..., u_n))$?
- **Jacobian matrix**: $\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial u_1} & \frac{\partial x_m}{\partial u_2} & \cdots & \frac{\partial x_m}{\partial u_n} \end{bmatrix}$
  - Answer to the fundamental question.
  - $m \times n$ matrix of all first-order partial derivatives of a vector-valued function.
- If $m = 1$: $\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$ for $f(x_1, x_2, ...x_n)$
- If $m = n$: $\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix}$
  - Most common scenario in change of variables.
  - **J** is a square matrix
  - **Jacobian**: $\frac{\partial(x_1, x_2, ..., x_n)}{\partial(u_1, u_2, ..., u_n)} = \det(\mathbf{J})$
  - The Jacobian is most applicable for a change of variables on iterated integrals.
  - Common Jacobians:
    - Cartesian plane-polar plane and Cartesian-cylindrical transformations
      - $(x, y) = (r\cos\theta, r\sin\theta)$ $\quad (r, \theta) = (\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right))$
      - $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ $\quad (r, \theta, z) = (\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right), z)$
      - $\frac{\partial(x, y)}{\partial(r, \theta)} = r$ $\quad \frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2}}$
    - Cartesian-spherical transformations
      - $(x, y, z) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$
      - $(\rho, \theta, \varphi) = \left(\sqrt{x^2 + y^2 + z^2}, \tan^{-1}\left(\frac{y}{x}\right), \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)\right)$
      - $\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = \rho^2\sin\varphi$ $\quad \frac{\partial(r, \theta, \varphi)}{\partial(x, y, z)} = \frac{1}{\rho^2\sin\varphi}$